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Short Communication

Analysis of propagation of waves of purely shear deformaton in a sandwich plate

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Abstract

This paper addresses propagation of waves of purely shear deformation in an unbounded sandwich plate composed of two identical isotropic skin plies and an isotropic core ply. To capture propagation of these waves, a three-dimensional formulation of the theory of elasticity ('refined theory') and a two-dimensional formulation of the sandwich plate 'elementary' theory are used as 'starting points' for the analysis. 'In-phase' and 'anti-phase' waves (with respect to in-plane deflections of skins) are considered independently of each other. For both types of motion, dispersion curves obtained by use of 'elementary' theory are compared with those obtained by use of the 'exact' theory (which involves the theory of elasticity in description of wave motion in a core ply). It is shown that the simplified models suggested are capable of giving an accurate description of propagating waves.

1. Introduction

This paper aims to complete the comparison of exact and simplified theories of wave motion in sandwich plates with and without heavy fluid loading presented in Ref. [1]. In this reference, the plane problem formulation has been explored, which reduces the stationary dynamics of a plate to its counterpart for a beam. However, a rectangular sandwich plate has a spectrum of resonant frequencies of vibrations in purely shear modes, which do not involve any transverse deflections,

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see Ref. [2]. The existence of this spectrum suggests that purely shear waves may propagate in an unbounded sandwich plate. These waves can be generated by, say, out-of-plane bending of stiffeners attached to the skins or by torsion of rods perpendicular to the plate and connected to one skin ply or to both of them. In a bounded plate, they can convey energy from an excitation zone to its remote parts and produce intensive transverse vibrations at the boundaries due to the modal coupling.

To capture such waves in an infinite plate, it is necessary to explore a spatial problem formulation in elastodynamics for a core ply (in the 'exact' theory) and to introduce standard assumptions, which define the purely shear deformation. Apparently, the heavy fluid loading has no influence on propagation of a purely shear wave as long as the fluid's viscosity is neglected. Analogously to the cases considered in Refs. [1–3], the simplified sandwich plate theories in two dimensions may also be formulated and their predictions may be compared with the exact solution. This is exactly the content of the present paper.

The paper is structured as follows. In Section 2, propagation of shear waves in sandwich plates is considered in the framework of a theory of elasticity. Due to the natural symmetry of a sandwich plate composition, two classes of wave motions ('in-phase' and 'anti-phase' ones) are analysed separately and two dispersion equations are derived. Elementary modelling of wave propagation for the same two classes of motions is presented in Section 3. The dispersion curves, which are obtained for a sandwich plate in the framework of these theories, are compared in Section 4. In Section 5, conclusions are presented.

2. Formulation of the problem within the framework of a theory of elasticity

Consider an unbounded sandwich plate consisting of two thin and relatively stiff isotropic plies (skins) and a soft isotropic core ply between them, as shown in Fig. 1. Since the 'potential' waves of coupled flexural and shear deformation have been systematically studied in both the exact and the simplified formulations in Ref. [1], here only the purely shear 'vortex'-type waves are considered. To investigate propagation of these waves, it is not possible to use the plane problem formulation (i.e., to consider cylindrical bending of a sandwich plate). However, an important

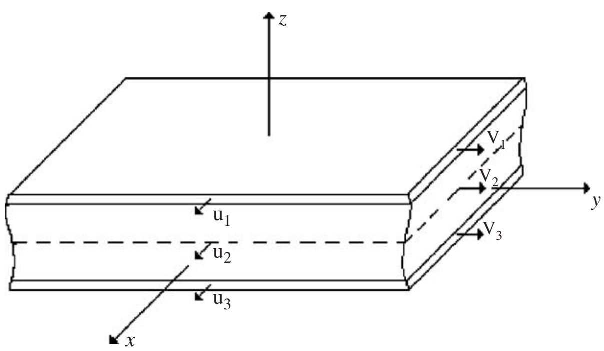


Fig. 1. Sandwich plate composition—in-plane displacement of plies.

simplification is introduced by the observation that the lateral displacement w in all three plies is absent [2] in this particular case.

As is well known [4], the mechanical properties of skin and core plies of sandwich plates used, for example, in naval or aerospace structures, are very different. Specifically, elastic and geometry parameters of skin plies considered individually are normally those of conventional thin plates. Elastic properties of the material of each skin ply are specified by densities ρ_k, $k = 1, 3$, Young moduli E_k, $k = 1, 3$, and Poisson coefficients $\nu_1 = \nu_3 = \nu$. Then, to study their in-plane deformation, it is sufficient to assume that the in-plane displacements of the mid-surfaces of skin plies are independent of transverse coordinate, $u_k = u_k(x, y, t)$, $v_k = v_k(x, y, t)$, $k = 1, 3$ (they are positive if co-directed with the coordinate axes shown in Fig. 1) and their out-of-plane displacement vanishes, $w_k \equiv 0$. Then, motions of skin plies are governed by the following equations:

$$\frac{E_k}{1-\nu^2}\frac{\partial^2 u_k}{\partial x^2} + \frac{E_k}{2(1-\nu)}\frac{\partial^2 v_k}{\partial x \partial y} + \frac{E_k}{2(1+\nu)}\frac{\partial^2 u_k}{\partial y^2} - \rho_k \frac{\partial^2 u_k}{\partial t^2} = -\frac{T_{xz}^{[k]}}{h_k}, \tag{1a}$$

$$\frac{E_k}{2(1-\nu)}\frac{\partial^2 u_k}{\partial x \partial y} + \frac{E_k}{2(1+\nu)}\frac{\partial^2 u_k}{\partial x^2} + \frac{E_k}{1-\nu^2}\frac{\partial^2 v_k}{\partial y^2} - \rho_k \frac{\partial^2 v_k}{\partial t^2} = -\frac{T_{yz}^{[k]}}{h_k}. \tag{1b}$$

Respectively, $T_{xz}^{[k]}(x, y, t)$ and $T_{yz}^{[k]}(x, y, t)$, $k = 1, 3$ are the interfacial shear stresses acting at skin plies from the core. In general, they may also contain external driving components, but as far as propagation of free waves is concerned an external loading is omitted.

The system of differential equations (1) is substantially simplified, when a purely shear deformation is addressed: $u_k(x, y, t) = u_k(y, t)$, $v_k(x, y, t) = v_k(x, t)$, $k = 1, 3$. The problem formulation is straightforwardly reduced to two uncoupled equations with respect to $u_k(y, t)$ and $v_k(x, t)$, $k = 1, 3$:

$$\frac{E_k}{2(1+\nu)}\frac{\partial^2 u_k}{\partial y^2} - \rho_k \frac{\partial^2 u_k}{\partial t^2} = -\frac{T_{xz}^{[k]}}{h_k}, \tag{2a}$$

$$\frac{E_k}{2(1+\nu)}\frac{\partial^2 v_k}{\partial x^2} - \rho_k \frac{\partial^2 v_k}{\partial t^2} = -\frac{T_{yz}^{[k]}}{h_k}. \tag{2b}$$

Due to the isotropy of skin plies, it is sufficient to solve any one of these two equations by setting, say, $v_k(x, t) = 0$ and $T_{yz}^{[k]}(x, t) = 0$, $k = 1, 3$. Therefore, a preferred direction of wave propagation along the y-axis is specified. As has been discussed, the core ply of a sandwich plate is much thicker and it is composed of material, which is much softer, than the skin plies. Thus, dynamics of a core ply should be described by the standard dynamic theory of elasticity, see for example Ref. [5]. Generally, the displacement field is formulated as

$$\vec{u} = \vec{\nabla}\varphi + \operatorname{rot}\vec{\psi}. \tag{3}$$

Here $\vec{u}(u_2, v_2, w_2)$ is a vector of displacements in a core ply, φ is a scalar potential and $\vec{\psi}(\psi_x, \psi_y, \psi_z)$ is a vector potential. The calibration condition is formulated as

$$\operatorname{div}\vec{\psi} = 0. \tag{4}$$

Inasmuch as a wave of purely shear deformation is concerned and the preferred direction of wave propagation is chosen, it is possible to adopt the following standard assumptions: $\varphi \equiv 0$, $\psi_x = 0$,

$\psi_y = \psi_0(y, z, t)$, $\psi_z = \psi_z(y, z, t)$. Then, the displacements field becomes

$$u_2(y, z, t) = \frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_0}{\partial z}, \quad v_2(y, z, t) \equiv 0, \quad w_2(y, z, t) \equiv 0. \tag{5}$$

The calibration condition (4) is re-written as

$$\frac{\partial \psi_z}{\partial z} = -\frac{\partial \psi_0}{\partial y}. \tag{6}$$

Thus, Lamé equations are reduced as follows:

$$\frac{\partial^2 \psi_0}{\partial y^2} + \frac{\partial^2 \psi_0}{\partial z^2} - \frac{1}{c_2^2}\frac{\partial^2 \psi_0}{\partial t^2} = 0, \tag{7a}$$

$$\frac{\partial^2 \psi_z}{\partial y^2} + \frac{\partial^2 \psi_z}{\partial z^2} - \frac{1}{c_2^2}\frac{\partial^2 \psi_z}{\partial t^2} = 0. \tag{7b}$$

Here $c_2^2 = E/2(1+v)\rho$ is the velocity of shear waves in the core material. The material density of a core ply, its Young module and Poisson coefficient are denoted as ρ, E and v, respectively.

Some elementary algebra gives the following formulation of the shear stresses ($G = E/2(1+v)$):

$$\tau_{xz}(y, z, t) = -G\left(\frac{\partial^2 \psi_0}{\partial z^2} - \frac{\partial^2 \psi_z}{\partial y \partial z}\right), \tag{8a}$$

$$\tau_{yz}(x, z, t) = 0. \tag{8b}$$

The normal stress also vanishes, $\sigma_z(x, z, t) \equiv 0$.

The system of differential equations (7) should be solved with the following compatibility conditions at the interfaces:

$$z = \frac{h}{2}: \quad u_2(y, z, t) = u_1(y, t), \tag{9a}$$

$$z = -\frac{h}{2}: \quad u_2(y, z, t) = u_3(y, t). \tag{9b}$$

The interfacial distributed forces involved in Eqs. (1) and (2) are formulated as

$$T_{xz}^{[1]}(y, t) = -\tau_{xz}\left(y, -\frac{h}{2}, t\right), \tag{10a}$$

$$T_{xz}^{[3]}(y, t) = \tau_{xz}\left(y, \frac{h}{2}, t\right). \tag{10b}$$

The following scaling is introduced: $y = \bar{y}h$, $z = \bar{z}h$, $u_j = \bar{u}_j h$, $w_j = \bar{w}_j h$, $j = 1, 2, 3$.

Propagation of a time-harmonic elastic wave in an unbounded plate is considered, so that

$$\bar{u}_j = U_j \exp(k\bar{y} - \mathrm{i}\omega t), \quad j = 1,\ 2,\ 3,$$

$$\psi_z = \Psi_z(\bar{z}) \exp(k\bar{y} - \mathrm{i}\omega t), \quad \psi_0 = \Psi_0(\bar{z}) \exp(k\bar{y} - \mathrm{i}\omega t). \tag{11}$$

Hereafter bars over non-dimensional variables are omitted, ω is a positive excitation frequency and k is, a priori, a complex wavenumber. Eq. (11) is substituted to Eqs. (7), (9) and the problem in elasticity for the core ply is formulated as

$$\frac{d^2\Psi_z}{dz^2} + \left[k^2 + \left(\frac{\omega h}{c_2}\right)^2\right]\Psi_z = 0, \tag{12a}$$

$$\frac{d^2\Psi_0}{dz^2} + \left[k^2 + \left(\frac{\omega h}{c_2}\right)^2\right]\Psi_0 = 0, \tag{12b}$$

$$z = \frac{1}{2}: \quad -k\Psi_z - \frac{d\Psi_0}{dz} = U_3, \tag{12c}$$

$$z = -\frac{1}{2}: \quad -k\Psi_z - \frac{d\Psi_0}{dz} = -U_1. \tag{12d}$$

It should be observed that there is no contradiction between the number of boundary conditions and the number of differential equations inasmuch as the calibration condition (6) must be taken into account in solving the system (12).

The symmetric composition of a sandwich plate ($h_1 = h_3$, $E_1 = E_3$, $\rho_1 = \rho_3$) is considered and therefore it is convenient to identify two uncoupled classes of linear wave motions and analyse them separately. One of them is related to shear vibrations, which occur in the 'anti-phase' mode for the skin plies, $U_1 = -U_3 = U_0$. Naturally, this class of motion is of vortex-type. In a sense, one may imagine clockwise and anti-clockwise vortices travelling in the 'upper' and the 'lower' skin plies, which induce twisting deformation in a core ply. In opposition, by letting $U_1 = U_3 = U_0$, a kind of 'in-phase' motions of skin plies of a sandwich plate is specified. This class of motion can be thought of as a conventional shear wave in the core ply coupled with similar waves in skins.

Consider the case of 'anti-phase' wave motions (i.e., $U_1 = -U_3 = U_0$). A general solution of Eqs. (12a) and (12b) is formulated as

$$\Psi_z(z) = B\sinh(\alpha z), \quad \Psi_0(z) = A\cosh(\alpha z), \quad \alpha^2 = -k^2 - \left(\frac{\omega h}{c_2}\right)^2. \tag{13}$$

The calibration condition acquires the elementary algebraic form

$$Ak = -B\alpha. \tag{14}$$

Respectively, the boundary conditions (12c) and (12d) are reduced to the single one

$$A\alpha\sinh\left(\frac{\alpha}{2}\right) + Bk\sinh\left(\frac{\alpha}{2}\right) = -U_0. \tag{15}$$

The solution of Eqs. (14) and (15) is $A = U_0\alpha/(\alpha^2 + k^2)\sinh(\alpha/2)$, $B = -kU_0/(\alpha^2 + k^2)\sinh(\alpha/2)$. The interfacial stresses then become

$$\tau_{xz}\left(\frac{h}{2}\right) = -G\alpha U_0\frac{\cosh(\alpha/2)}{\sinh(\alpha/2)}. \tag{16}$$

The dispersion equation is

$$k^2 = -\frac{2(1+\nu)\rho_1\omega^2 h^2}{E_1} + \frac{E}{E_1}\frac{h}{h_1}\alpha\frac{\cosh(\alpha/2)}{\sinh(\alpha/2)}. \tag{17}$$

A solution in the case of 'in-phase' motions ($U_1 = U_3 = U_0$) is obtained in a similar way. Elastic potentials are defined as

$$\Psi_z(z) = \tilde{B}\cosh(\alpha z), \quad \Psi_0(z) = \tilde{A}\sinh(\alpha z), \quad \alpha^2 = -k^2 - \left(\frac{\omega h}{c_2}\right)^2. \tag{18}$$

The calibration condition preserves the form (14). Respectively, the boundary conditions (12c) and (12d) are now reduced to

$$\tilde{A}\alpha\cosh\left(\frac{\alpha}{2}\right) + \tilde{B}k\cosh\left(\frac{\alpha}{2}\right) = -U_0. \tag{19}$$

The solution of Eqs. (14) and (19) is $\tilde{A} = U_0\alpha/(\alpha^2+k^2)\cosh(\alpha/2)$, $\tilde{B} = -kU_0/(\alpha^2+k^2)\cosh(\alpha/2)$. The interfacial stresses then become

$$\tau_{xz}\left(\frac{1}{2}\right) = -G\alpha U_0\frac{\sinh(\alpha/2)}{\cosh(\alpha/2)}. \tag{20}$$

The dispersion equation for this type of motion is

$$k^2 = -\frac{2(1+\nu)\rho_1\omega^2 h^2}{E_1} + \frac{E}{E_1}\frac{h}{h_1}\alpha\frac{\sinh(\alpha/2)}{\cosh(\alpha/2)}. \tag{21}$$

The dispersion equations (17) and (21) have infinitely large number of roots, which are either purely real, or purely imaginary, or complex. In principle, various methods may be used to find these roots, depending on available software and computing facilities. In this paper, the propagation of waves in the sandwich structures used in shipbuilding or aerospace industries is addressed. From the practical viewpoint, it is most important to find the roots, which define propagating wave in the not-too-high frequency range. Then, the first roots are not too large and the transcendent dispersion equations may conveniently be transformed to a simple polynomial in k^2 form by expanding hyper-trigonometric functions into power series and elementary algebraic manipulations. The order of this dispersion equation is controlled by a number of terms retained in power series. For each particular value of a frequency parameter, all roots of this approximate polynomial equation are readily found numerically by use of, for example, symbolic manipulator *Mathematica* [6]. They are then used one by one as the 'initial guess' to search numerically for the roots of the original dispersion equations. This procedure gives 'refined' values of wavenumbers, and the accuracy of a polynomial approximation of dispersion equations is readily accessed.

This 'exact' or 'refined' solution of the problem of wave propagation in a sandwich plate is used hereafter to check the validity of simplified models. A set of simplified theories, which may be used to describe first branches of dispersion curves, is formulated in the following section. The existence of an infinite number of solutions of dispersion equations (17) and (21) suggests that, at the boundaries of a sandwich plate, considered in the framework of a 'refined' theory, it is necessary to formulate conditions being continuously held along the thickness of a core ply, i.e.

for $-\frac{1}{2}<z<\frac{1}{2}$. As shown in Section 4, these solutions of dispersion equations (17) and (21) are present in the considered frequency range evanescent waves. They do not contribute to the energy transportation, but play a crucial role in formation of a boundary layer in the vicinity of boundaries. Various 'high-order' sandwich plate theories are meant, in effect, to utilise certain modes predicted by the 'refined' theory and therefore introduce special boundary conditions. Their asymptotical correctness (with respect to shear wave modelling) may be easily assessed by comparison of dispersion curves predicted by them with those given by formulas (17) and (21). However, this analysis lies beyond the scope of the present paper.

3. Elementary modelling of propagation of waves in sandwich plates

The elementary theory of flexural and shear vibrations of sandwich plates used in Refs. [1–3] is in a certain way a generalisation of the classical Timoshenko theory for homogeneous plates [7]. A vector of shear angle $\vec{\vartheta}$ between skin plies is introduced as an independent variable in addition to a lateral deflection of the whole package of three plies w. All plies are assumed to be isotropic and the following non-dimensional parameters describe the internal structure of a sandwich plate: $\varepsilon = h_1/h$ as a thickness parameter, $\delta = \rho/\rho_1$ as a density parameter, $\gamma = E/E_1$ as a stiffness parameter. In the framework of this theory (see also Ref. [2]), the deformation of a sandwich plate element is governed by three independent scalar variables: a displacement of the mid-surface of the whole element w (which is the same for all plies), a shear angle ϑ_x about the y-axis and a shear angle ϑ_y about the x-axis, see Fig. 2:

$$-m\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^4 w}{\partial t^2 \partial x^2} + I_1\frac{\partial^4 w}{\partial t^2 \partial y^2} - \frac{\partial^2}{\partial x^2}\left(D_1\frac{\partial^2 w}{\partial x^2}\right) - v\frac{\partial^2}{\partial x^2}\left(D_1\frac{\partial^2 w}{\partial y^2}\right) - v\frac{\partial^2}{\partial y^2}\left(D_1\frac{\partial^2 w}{\partial x^2}\right)$$
$$-\frac{\partial^2}{\partial y^2}\left(D_1\frac{\partial^2 w}{\partial y^2}\right) - 4\frac{\partial^2}{\partial x \partial y}\left(C_1\frac{\partial^2 w}{\partial x \partial y}\right) + \frac{\partial}{\partial x}(\Gamma\vartheta_x) + \frac{\partial}{\partial y}(\Gamma\vartheta_y) + \frac{\partial}{\partial x}\left(\Gamma\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial w}{\partial y}\right) = 0,$$

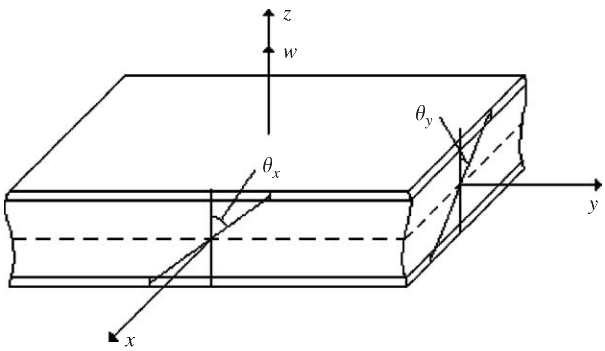


Fig. 2. Sandwich plate composition—shear angles.

$$-I_2\frac{\partial^2\vartheta_x}{\partial t^2}+D_2\frac{\partial^2\vartheta_x}{\partial x^2}+\nu D_2\frac{\partial^2\vartheta_y}{\partial x\partial y}+C_2\left(\frac{\partial^2\vartheta_y}{\partial x\partial y}+\frac{\partial^2\vartheta_x}{\partial y^2}\right)-\Gamma\left(\vartheta_x+\frac{\partial w}{\partial x}\right)=0,$$

$$-I_2\frac{\partial^2\vartheta_y}{\partial t^2}+D_2\frac{\partial^2\vartheta_y}{\partial y^2}+\nu D_2\frac{\partial^2\vartheta_x}{\partial x\partial y}+C_2\left(\frac{\partial^2\vartheta_x}{\partial x\partial y}+\frac{\partial^2\vartheta_y}{\partial x^2}\right)-\Gamma\left(\vartheta_y+\frac{\partial w}{\partial y}\right)=0. \tag{22}$$

The effective elastic parameters are

$$D_1=\frac{E_1h_1^3}{12(1-\nu^2)}\left(2+\frac{\gamma}{\varepsilon^3}\right),\quad C_1=\frac{E_1h_1^3}{12(1-\nu^2)}\left(2+\frac{\gamma}{\varepsilon^3}\right)\frac{1-\nu}{2},$$
$$D_2=\frac{E_1h_1^3}{2(1-\nu^2)}\left(1+\frac{1}{\varepsilon}\right)^2,\quad C_2=\frac{E_1h_1^3}{2(1-\nu^2)}\left(1+\frac{1}{\varepsilon}\right)^2\frac{1-\nu}{2},$$
$$\Gamma=\frac{E_1h_1}{(1-\nu^2)}\frac{1-\nu}{2}\varepsilon\gamma\left(1+\frac{1}{\varepsilon}\right)^2,$$
$$I_1=\frac{\rho_1h_1^3}{12}\left(2+\frac{\delta}{\varepsilon^3}\right),\quad I_2=\frac{\rho_1h_1^3}{2}\left(1+\frac{1}{\varepsilon}\right)^2,\quad m=\rho_1h_1\left(2+\frac{\delta}{\varepsilon}\right). \tag{23}$$

In the case of a purely shear wave, $w=0$, so that

$$\frac{\partial\vartheta_x}{\partial x}+\frac{\partial\vartheta_y}{\partial y}=0, \tag{24a}$$

$$-I_2\frac{\partial^2\vartheta_x}{\partial t^2}+D_2\frac{\partial^2\vartheta_x}{\partial x^2}+\nu D_2\frac{\partial^2\vartheta_y}{\partial x\partial y}+C_2\left(\frac{\partial^2\vartheta_y}{\partial x\partial y}+\frac{\partial^2\vartheta_x}{\partial y^2}\right)-\Gamma\vartheta_x=0, \tag{24b}$$

$$-I_2\frac{\partial^2\vartheta_y}{\partial t^2}+D_2\frac{\partial^2\vartheta_y}{\partial y^2}+\nu D_2\frac{\partial^2\vartheta_x}{\partial x\partial y}+C_2\left(\frac{\partial^2\vartheta_x}{\partial x\partial y}+\frac{\partial^2\vartheta_y}{\partial x^2}\right)-\Gamma\vartheta_y=0. \tag{24c}$$

The 'Airy'-type function $\Theta(x,y,t)$ is introduced as $\vartheta_x=-\partial\Theta/\partial y$, $\vartheta_y=\partial\Theta/\partial x$ to satisfy Eq. (24a) automatically. Furthermore, the preferred direction of wave propagation is chosen as $\Theta(x,y,t)=\Theta(y,t)$. Thus, Eqs. (24b) and (24c) are reduced to a single one:

$$I_2\frac{\partial^3\Theta}{\partial y\partial t^2}-C_2\frac{\partial^3\Theta}{\partial y^3}+\Gamma\frac{\partial\Theta}{\partial y}=0. \tag{25}$$

Standard substitution $\Theta=\hat{\Theta}\exp(ky-\mathrm{i}\omega t)$ leads to an elementary formula, which defines a wavenumber as

$$k=\sqrt{-\frac{2(1+\nu)\rho_1\omega^2h^2}{E_1}+2\frac{\gamma}{\varepsilon}}. \tag{26}$$

In this theory, only 'anti-phase' in-plane motions of skin plies are considered and this root is expected to be reasonably close to the first root of the transcendent exact equation (17) in the practically meaningful range of parameters. This aspect is addressed in Section 3.

Another type of motion of skin plies should also be considered, which involves their simultaneous 'in-phase' in-plane displacements. It has already been introduced in the previous section in the framework of a theory of elasticity. Similarly to the elementary theory of a sandwich plate, which models its 'anti-phase' motions, a separate simple model may be suggested in this case. This model is already provided by Eq. (2), which should be applied with appropriately averaged stiffness and inertial parameters. Apparently, the interfacial stresses must be set to zero. Then the equation of motion is

$$D_{\rm eq}\frac{\partial^2 u}{\partial y^2} - m_{\rm eq}\frac{\partial^2 u}{\partial t^2} = 0. \tag{27}$$

Here, an equivalent stiffness and an equivalent inertia are straightforwardly defined as

$$D_{\rm eq} = 2G_1h_1 + Gh, \quad m_{\rm eq} = 2\rho_1 h_1 + \rho h.$$

A solution of Eq. (27) is sought as

$$u_0(y,t) = U_0 \exp(ky - {\rm i}\omega t).$$

Then the root of dispersion equation is

$$k = {\rm i}\sqrt{\frac{2\rho_1\omega^2 h^2(1+\nu)}{E_1}\frac{\left(1+\frac{1}{2}\frac{\rho}{\rho_1}\frac{h}{h_1}\right)}{\left(1+\frac{1}{2}\frac{E}{E_1}\frac{h}{h_1}\right)}}. \tag{28}$$

It defines the branch of dispersion curves, which should agree with the first one, obtained from Eq. (21) formulated by an exact solution of the theory of elasticity.

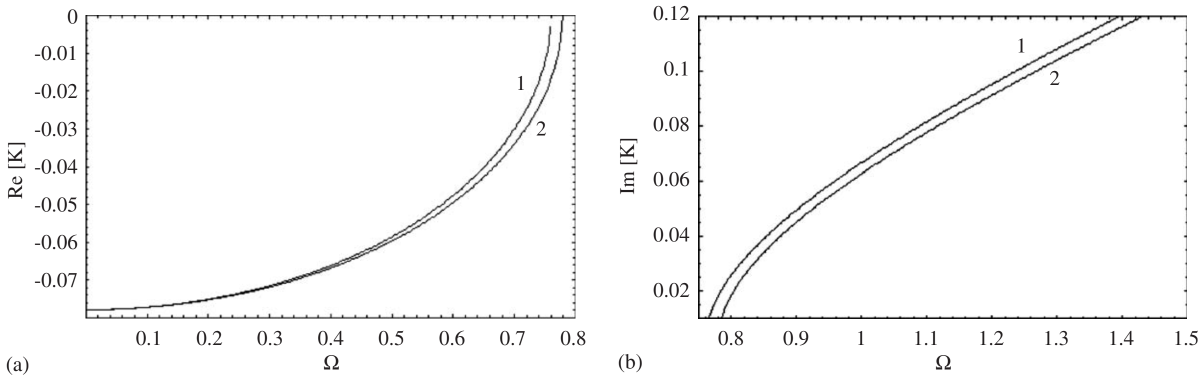


Fig. 3. Dispersion curves (a—real parts; b—imaginary parts) for the first 'anti-phase' wave. The parameters of sandwich plate composition: $\varepsilon = 0.33$, $\gamma = 0.001$, $\delta = 0.1$.

4. Dispersion curves

4.1. 'Anti-phase' waves

Compare the results obtained for 'anti-phase' in-plane wave motions of skin plies. In Fig. 3, the lowest roots of dispersion equations (17) and (26) are shown versus the frequency parameter $\Omega \equiv \omega h/c_2$ for $\varepsilon = 0.33$, $\gamma = 0.001$, $\delta = 0.1$, $v = 0.3$. Curves 1 are plotted after 'refined' theory, curves 2 are plotted after 'elementary' theory. As is seen in Fig. 3a, the roots of these dispersion equations are purely real in the low-frequency range and they describe an evanescent wave. The cut-on frequency parameter is predicted by these theories as $\Omega_{\text{cut-on}} \approx 0.78$ (elementary theory) and $\Omega_{\text{cut-on}} \approx 0.76$ (exact theory), i.e., there is a difference in 2.6%. In Fig. 3b, the same roots are presented as they transform to purely imaginary ones, which describe travelling waves. The agreement between the 'refined' and the 'elementary' theories at low frequencies is very good, see Fig. 3a, and it is reasonably good at somewhat higher frequencies, see Fig. 3b. It is readily explained in a closer inspection into the structure of Eq. (17). Indeed, the transcendent functions involved in this formula may conveniently be expanded into power series on the parameter α regarded as being small. As long as a single-term expansion is sufficiently accurate ($\sinh(\alpha/2) \approx \alpha/2$, $\cosh(\alpha/2) \approx 1$), the exact solution recovers the dispersion equation (26). This result proves that the elementary theory of 'anti-phase' motions is asymptotically correct. Apparently, the error grows with the growth in a frequency parameter, so that the propagating wave is not described sufficiently accurately already in the vicinity of a cut-on frequency. The exact theory predicts infinitely many branches of dispersion curves, but in the low-frequency range all of them are complex-valued. The curve in Fig. 4a displays the real part of the 'second' root (there also exist its complex conjugate) of dispersion equation (17), the curve in Fig. 4b displays its imaginary part.

The agreement between the exact theory (17) and the elementary theory (26) is also good for the smaller values of density ratio and stiffness ratio. However, for a thicker core, the accuracy of the elementary theory becomes slightly less satisfactory. It is seen in Fig. 5, plotted for $\varepsilon = 0.15$, $\gamma = 0.001$, $\delta = 0.1$, $v = 0.3$ in the same way as Fig. 3. The cut-on frequency is predicted by the exact theory as $\Omega_{\text{cut-on}} \approx 1.095$ and by the elementary theory as $\Omega_{\text{cut-on}} \approx 1.158$ (the error is

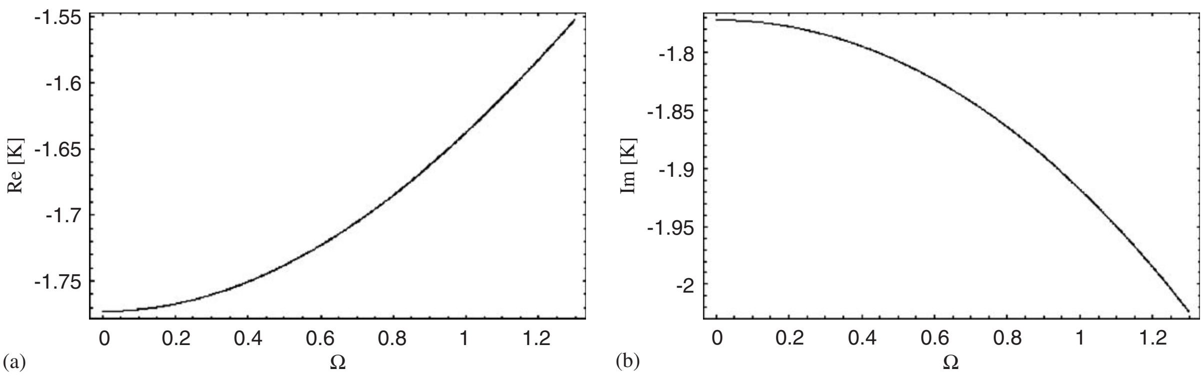


Fig. 4. Dispersion curves (a—real parts; b—imaginary parts) for the second 'anti-phase' wave. The parameters of sandwich plate composition: $\varepsilon = 0.33$, $\gamma = 0.001$, $\delta = 0.1$, $v = 0.3$.

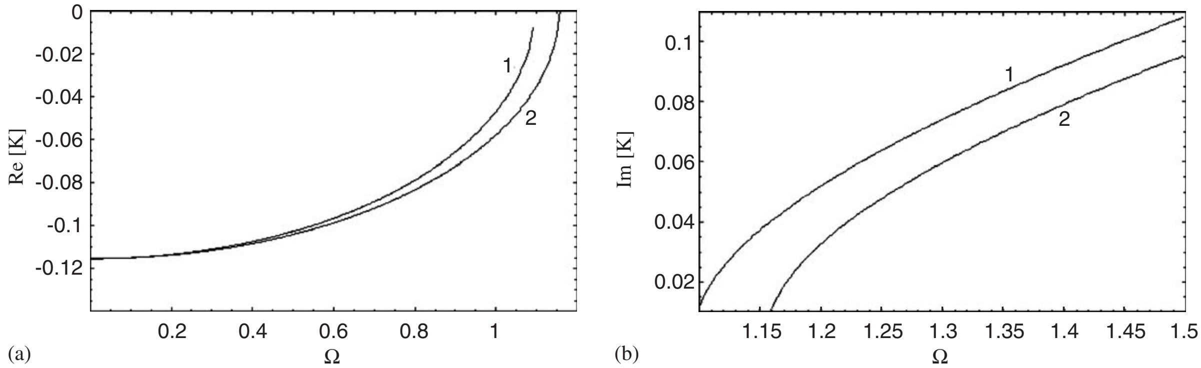


Fig. 5. Dispersion curves (a—real parts; b—imaginary parts) for the first 'anti-phase' wave. The parameters of sandwich plate composition: $\varepsilon = 0.15$, $\gamma = 0.001$, $\delta = 0.1$.

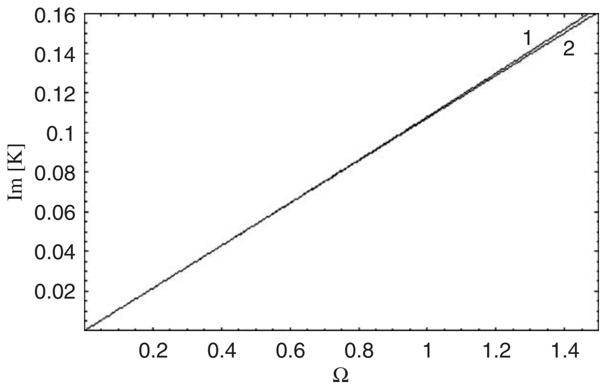


Fig. 6. Dispersion curves for the 'in-phase' propagating wave. The parameters of sandwich plate composition: $\varepsilon = 0.33$, $\gamma = 0.001$, $\delta = 0.1$.

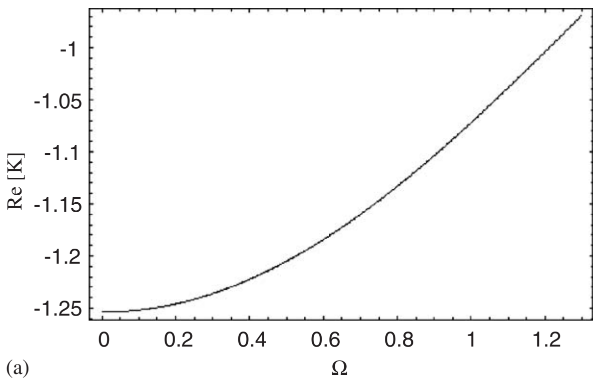
5.8%). The magnitude of the wavenumber of a propagating wave is predicted less inaccurately by the elementary theory in this case, than in the previous one. Naturally, at low frequencies this theory remains asymptotically correct, see Fig. 5a. This 'breakdown' of an elementary theory is easily explained by the observation that formation of a standing wave across a thick core ply quickly deviates from the linear function, 'embedded' in the derivation of the elementary theory. On the other hand, it should be pointed out that, as seen from formula (26), the growth in parameter ε (which damages the validity of the elementary theory) results in an increase in the magnitude of a cut-on frequency, so that this 'anti-phase' wave does not propagate in practically meaningful low-frequency range.

4.2. 'In-phase' waves

In Fig. 6, dispersion curves 1 (exact theory, Eq. (21)) and 2 (elementary theory, formula (28)) showing a dependence of the non-dimensional wavenumber k on the frequency parameter

$\Omega \equiv \omega h/c_2$ are plotted for $\varepsilon = 0.33$, $\gamma = 0.001$, $\delta = 0.1$, $v = 0.3$. The agreement between the 'refined' and the 'elementary' theories is very good indeed. It is readily explained in a closer inspection into the structure of Eq. (21). The transcendent functions involved in this formula may conveniently be expanded into power series on the parameter α regarded as being small. As long as a single term expansion is sufficiently accurate ($\sinh(\alpha/2) \approx \alpha/2$, $\cosh(\alpha/2) \approx 1$), the exact solution recovers the dispersion equation (28). This result proves that the elementary theory of 'in-phase' motions is also asymptotically correct.

Curves shown in Fig. 6 display a dependence of the first wavenumber (the one with the minimal magnitude) on the frequency parameter Ω. The next branch predicted by the 'exact' theory describes an attenuated wave with a high decay rate. The real and imaginary parts of this wavenumber are presented in Fig. 7a and b. This wave cuts on at a very high (from the practical point of view) frequency. Unlike the case of 'in-phase' waves, the accuracy of the elementary theory remains very satisfactory for a thicker core. It suggests that the deformation of a core ply between skins, which move in phase with each other with the same amplitude, remains uniform at relatively high frequencies. This is fairly obvious intuitively. The graph in Fig. 8 is plotted for the



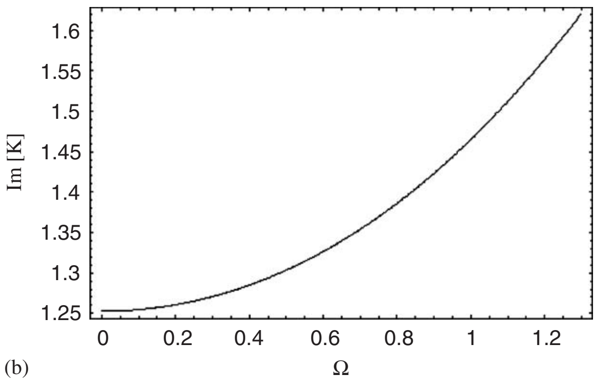


Fig. 7. Dispersion curves (a—real parts; b—imaginary parts) for the second 'in-phase' wave. The parameters of sandwich plate composition: $\varepsilon = 0.33$, $\gamma = 0.001$, $\delta = 0.1$, $v = 0.3$.

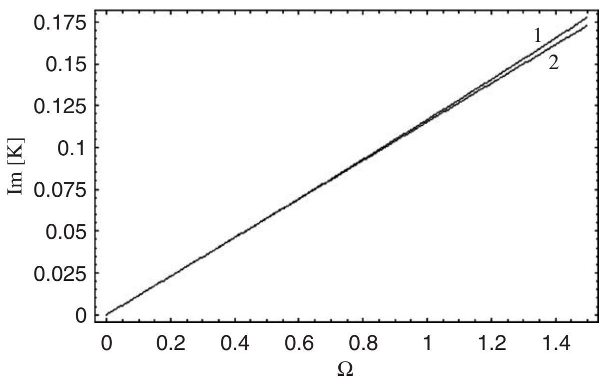


Fig. 8. Dispersion curves for the 'in-phase' propagating wave. The parameters of sandwich plate composition: $\varepsilon = 0.15$, $\gamma = 0.001$, $\delta = 0.1$.

following set of parameters of a sandwich plate composition: $\varepsilon = 0.15$, $\gamma = 0.001$, $\delta = 0.1$, $v = 0.3$. The elementary theory (curve 2) gives excellent predictions in the whole frequency band considered.

5. Conclusions

Two 'elementary' theories are suggested to describe the propagation of purely shear waves in sandwich plates. Comparison of dispersion curves obtained by solving characteristic equations derived from these theories with dispersion curves obtained in the 'exact' problem formulation shows that these theories may be reliably used to assess the wave guide properties of a sandwich plate in the frequency range of practical interest.

Acknowledgements

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